

On a class of poverty measures based on weighted poverty gaps

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Abstract

We consider a class of poverty measures based on ranks that generalizes, among other measures, the class of linear indices discussed by Hagenaars (1987) and the class of equally distributed equivalent (EDE) poverty gaps considered by Duclos and Gregoire (2002). This class, introduced by Sordo, Ramos and Ramos (2007), is closely related to the TIP (Three I's of poverty) curve, a graphical device used to describe distributional poverty (Jenkins and Lambert, 1997). In this paper, we study the consistency of this class of measures with the most commonly accepted axioms for poverty measures and illustrate, with Spanish real data, the connection between stochastic dominance of TIP areas and orderings of income distributions according to these measures.

Keywords: poverty measure, TIP curve, poverty axiom.

1 Introduction

Having good measures of poverty is essential to address the poverty problem derived from the global economic crisis. From the seminal paper of Sen (1976), numerous measures of poverty have been proposed and some of them guide institutions' development policies. However, in practice, the appropriate measure to use to evaluate poverty continues to be a subject of academic debate and studio. There are many detailed reviews of the poverty measure construction issue, including Sen (1976, 1981), Foster (1984), Hagenaaars (1987), Seidl (1988), Chakravarty (1990), Ravallion (1994), Foster and Sen (1997) and Zheng (1997, 2000).

The first issue when measuring poverty in Sen's approach is to identify the poor among the total population. This can be made by using a poverty line, z ($z > 0$), which separates the population into the poor and nonpoor subgroups: a member of the population is poor if its income is below the poverty line. The second issue is the construction of an index to measure the intensity of their poverty. Thus, if D denotes the set of all income random variables, a poverty measure is a function $P : D \times R^+ \rightarrow R^+$ whose value $P(X, z)$ indicates the degree of poverty for a given income random variable X and a poverty line z .

Poverty measures, as other economic indices, are judged according to the axioms that they satisfy. Different classes of poverty measures describe different aspects of poverty and represent different schools of thought (we summarize below a set of axioms which have achieved significant consensus in the literature). According to their structure, many of the poverty measures that have been proposed can be classified in two classes: the class of additive poverty measures and the class of poverty measures based on ranks. The first one includes, among others, the popular headcount and poverty gap indices, the measures given by Watts (1968), Chakravarty (1983) and Foster, Greer and Thorbecke (1984). The second one, with which we are concerned in this paper, contains the poverty measure proposed by Sen (1976) and its variations, as well as the linear indices given by Hagenaaars (1987) and Duclos and Gregoire (2002). Specifically, we focus on a family of rank-based poverty measures, C , that generalizes, among other measures, the class of linear indices discussed by Hagenaaars (1987) and the class of equally distributed equivalent (EDE) poverty gaps considered by Duclos and Gregoire (2002). This class, introduced by Sordo, Ramos and Ramos (2007), is based on weighted averages of deviations of quantiles from z , a construction method that has sev-

eral precedents in the economic and statistical literature (see Mehran (1976), Yaari (1988), Ramos and Sordo (2003) and Sordo, Ramos and Ramos (2007)). Indices of C are closely related to the TIP (Three I's of poverty) curve, a graphical device used to describe distributional poverty (see Jenkins and Lambert, 1997). However, despite the class C and its subclasses have been completely characterized by first and higher-degree stochastic dominance of TIP curves and areas (see Sordo, Ramos and Ramos (2007) and Sordo and Ramos (2011)), a good understanding of its properties is still required. In particular, one aspect of the acceptability of this class of poverty measures, which has not yet been studied in depth, is its consistency with the poverty axioms. One purpose of this paper is to address this issue. Another purpose is to illustrate with real data the connection between stochastic dominance of TIP areas and orderings of income distributions according to measures of C .

To that end, we recall in Section 2 the class C of poverty measures. Each members of C is defined by means of a Riemann-Stieltjes integral, which allows simultaneous treatment of the purely discrete and absolutely continuous income random variables. In order to give subsequent interpretations, we concrete in Section 3 the expression that these measures take for discrete set of incomes. Section 4 describes some particular members of the class C . In Section 5 we study the consistency of members of C with the poverty axioms. In Section 6, we review the connection between the TIP curve and the class C . Finally, to illustrate this connection, we include in Section 7 a comparative study between poverty of several autonomous communities in Spain.

2 A class of poverty measures

Let X be a non-negative income random variable with a cumulative distribution function (*cdf*) F and let F^{-1} be the right continuous quantile function of F , which is defined by

$$F^{-1}(t) = \{x : F(x) \leq t\}, \quad t \in [0, 1]. \quad (1)$$

Throughout this work, we use the term “income” to signify a measure of individual welfare (not necessarily money income). Some poverty indices require strictly positive values of X so, for expositional simplicity, we may

assume that is always the case¹.

A poverty line, z ($z > 0$), separates the population into the poor and nonpoor subgroups and a member of the population is poor if its income is below the poverty line. The proportion of poor people is denoted as r_z , that is

$$r_z = \{F(x) : x < z\}. \quad (2)$$

We need the concept of quantiles censored at a poverty line z . Let $X_z^* = \{X, z\}$ be the random variable, X , censored at z , with distribution function, F_z . Its corresponding quantile function is F_z^{-1} , where

$$F_z^{-1}(t) = \begin{cases} F^{-1}(t), & \text{if } t < r_z \\ z, & \text{if } t \geq r_z \end{cases}, \quad (3)$$

for all $t \in [0, 1]$. Censored quantiles are, therefore, just the incomes $F^{-1}(t)$ for those in poverty and z for those whose income exceeds the poverty line.

The poverty gap associated with income $F^{-1}(t)$ is defined as $(z - F_z^{-1}(t))$, and it is the shortfall of income $F^{-1}(t)$ from the poverty line. When income $F^{-1}(t)$ exceeds the poverty line, the poverty gap equals zero. The normalized poverty gaps are defined as

$$\frac{z - F_z^{-1}(t)}{z}.$$

Sordo, Ramos and Ramos (2007) and Sordo and Ramos (2011) have considered a class C of linear poverty measures given by the following functional form:

$$I_X(\Phi, z) = \int_0^1 \left(\frac{z - F_z^{-1}(t)}{z} \right) d\Phi(t), \quad (4)$$

where Φ is any continuous probability distribution function with support in $[0, 1]$ (the integral is interpreted in the sense of Riemann-Stieltjes). The class C is similar to the class of linear inequality measures proposed by Mehran (1976) for inequality indices, by Yaari (1988) for social welfare indices and for Ramos and Sordo (2003) for dispersion indices in statistics. Observe that $I_X(\Phi, z)$ represents a weighted average of relative deviations of the censored incomes $F_z^{-1}(t)$ from the poverty line z . We note that a number of well-known poverty measures are the result of different ways of weighting these

¹It is reasonable to expect variables such as consumption or expenditures to be strictly positive. This assumption is less natural for other indicators, such as income, for which capital losses can generate negative values.

deviations. In particular, we will show in Section 4 that the linear poverty indices discussed by Hagenaars (1987) in the context of finite populations and the EDE poverty gaps considered by Duclos and Gregoire (2002) can be expressed as particular members of \mathcal{C} .

If we denote by H_z the function

$$H_z(t) = \frac{F_z^{-1}(t)}{z} = \begin{cases} \frac{F^{-1}(t)}{z} & \text{if } 0 \leq t < r_z \\ 1 & \text{if } r_z \leq t \leq 1, \end{cases} \quad (5)$$

then $I_X(\Phi, z)$ can be rewritten as

$$I_X(\Phi, z) = \int_0^{r_z} (1 - H_z(t)) d\Phi(t). \quad (6)$$

Observe that $I_X(\Phi, z)$ is well defined because Φ is continuous over the interval $[0, 1]$ and $(1 - H_z(t))$ is of bounded variation over the interval $[0, 1]$ (Apostol, 1977). Since $0 \leq 1 - H_z(t) \leq 1$ for $0 \leq t \leq 1$, $d\Phi(t) \geq 0$ and $\Phi(0) = 0$ we have

$$0 \leq I_X(\Phi, z) = \int_0^{r_z} (1 - H_z(t)) d\Phi(t) \leq \int_0^{r_z} d\Phi(t) = \Phi(r_z), \quad (7)$$

regardless of the poor's income. Moreover, the measure $I_X(\Phi, z)$ verifies the following convenient property.

Proposition 1 *Let F be the distribution function of a non-negative random variable X . For a given poverty line $z > 0$, $I_X(\Phi, z)$ is zero if the fraction of the poor is zero and takes on its maximal value if every poor has income zero.*

Proof. If the proportion of the poor is zero, $r_z = 0$, it is clear from (7) that the poverty measure $I_X(\Phi, z) = 0$. On the other hand, if every poor has income zero then $F(x) = r_z$ for $0 \leq x < z$, which implies that $F^{-1}(t) = 0$ for $0 \leq t < r_z$. Then,

$$H_z(t) = \begin{cases} 0, & \text{if } 0 \leq t < r_z, \\ 1, & \text{if } r_z \leq t \leq 1. \end{cases} \quad (8)$$

Integration by parts of

$$I_X(\Phi, z) = \int_0^{r_z} (1 - H_z(t)) d\Phi(t)$$

gives

$$\Phi(r_z) (1 - H_z(r_z)) - \Phi(0) (1 - H_z(0)) - \int_0^{r_z} \Phi(t) d(1 - H_z(t)). \quad (9)$$

Using that $H_z(r_z) = 1$ and $\Phi(0) = 0$ (Φ is a continuous distribution function with support in $[0, 1]$), we see that (9) is the same as

$$\int_0^{r_z} \Phi(t) dH_z(t). \quad (10)$$

From (8) and (10) it follows that

$$I_X(\Phi, z) = \Phi(r_z), \quad (11)$$

and from (7) and (11) it follows that $I_X(\Phi, z)$ takes on its maximal value if every poor has income zero. ■

Remark 2 *Since Φ is the distribution function of a random variable with support contained in $[0, 1]$, from (7) it follows that $0 \leq I_X(\Phi, z) \leq 1$.*

3 The case of a finite population

The Riemann-Stieltjes integral allows simultaneous treatment of the discrete and absolutely continuous case. However, in order to give subsequent interpretations, we concrete the expression that $I_X(\Phi, z)$ has when X is the discrete variable of a finite set of incomes. Our first result considers the case of a discrete random variable X that takes α different values (x_1, \dots, x_α) with probability mass function $(\frac{n_1}{n}, \frac{n_2}{n}, \dots, \frac{n_\alpha}{n})$, $\sum_{i=1}^{\alpha} n_i = n$. This model corresponds to a finite population where there are n_1 individuals with income x_1 , n_2 individuals with income x_2 and so on.

Theorem 3 *Let $0 \leq x_1 < x_2 < \dots < x_\alpha$ denote the distribution of income among n income-receiving units with respective mass $\frac{n_1}{n}, \frac{n_2}{n}, \dots, \frac{n_\alpha}{n}$*

($\sum_{i=0}^{\alpha} n_i = n$, $n_0 = 0$). Let $z > 0$ be a poverty line and let l be the integer such that $x_l < z \leq x_{l+1}$. Then

$$I_X(\Phi, z) = \sum_{k=1}^l \left[\Phi \left(\sum_{i=0}^k \frac{n_i}{n} \right) - \Phi \left(\sum_{i=0}^{k-1} \frac{n_i}{n} \right) \right] \left(\frac{z - x_k}{z} \right).$$

Proof. Since X is discrete, $H_z(t)$ is a step function with discontinuities at the points

$$t_k = \sum_{i=1}^k \frac{n_i}{n}, \quad k = 1, 2, \dots, l,$$

where $\sum_{i=1}^l \frac{n_i}{n} = r_z$ (i.e., $x_l < z \leq x_{l+1}$). The magnitude of the jump at these points is given by the expression

$$\Delta(t_k) = \begin{cases} \left(\frac{x_{k+1} - x_k}{z} \right), & \text{if } k = 1, 2, \dots, l-1, \\ \left(\frac{z - x_l}{z} \right), & \text{if } k = l. \end{cases}$$

Since the Riemann-Stieltjes integral becomes a finite sum when the integrand is a step function (Apostol, 1977), we can rewrite the expression of $I_X(\Phi, z)$ given by (10) in the following manner

$$\begin{aligned} I_X(\Phi, z) &= \sum_{k=1}^{l-1} \Phi \left(\sum_{i=0}^k \frac{n_i}{n} \right) \left(\frac{x_{k+1} - x_k}{z} \right) + \Phi \left(\sum_{i=0}^l \frac{n_i}{n} \right) \left(\frac{z - x_l}{z} \right) \\ &= \sum_{k=1}^{l-1} \Phi \left(\sum_{i=0}^k \frac{n_i}{n} \right) \left(\frac{z - x_k}{z} - \frac{z - x_{k+1}}{z} \right) + \Phi \left(\sum_{i=0}^l \frac{n_i}{n} \right) \left(\frac{z - x_l}{z} \right) \\ &= \sum_{k=1}^l \left[\Phi \left(\sum_{i=0}^k \frac{n_i}{n} \right) - \Phi \left(\sum_{i=0}^{k-1} \frac{n_i}{n} \right) \right] \left(\frac{z - x_k}{z} \right). \end{aligned}$$

■

As it will be shown in the next section, the following corollary makes explicit the relationship between (4) and some classical poverty indices originally defined for finite populations.

Corollary 4 Let $0 \leq x_1 \leq x_2 \leq \dots \leq x_n$ denote the distribution of income among n income-receiving units. Let $z > 0$ be a poverty line and let q be the number of poor. Then,

$$I_X(\Phi, z) = \sum_{i=1}^q \left[\Phi\left(\frac{i}{n}\right) - \Phi\left(\frac{i-1}{n}\right) \right] \left(\frac{z - x_i}{z} \right). \quad (12)$$

Proof. Suppose that there are α different incomes between the n income-receiving units which denote $x'_1 < x'_2 < \dots < x'_\alpha$, with respective mass $\frac{n_1}{n}, \frac{n_2}{n}, \dots, \frac{n_\alpha}{n}$, where $\sum_{i=0}^{\alpha} n_i = n$ and $n_0 = 0$. Denote by l the integer such that $x'_l < z \leq x'_{l+1}$. Then, by Theorem 3

$$I_X(\Phi, z) = \sum_{j=1}^l \left[\Phi\left(\sum_{i=0}^j \frac{n_i}{n}\right) - \Phi\left(\sum_{i=0}^{j-1} \frac{n_i}{n}\right) \right] \left(\frac{z - x'_j}{z} \right). \quad (13)$$

It is not difficult to show that (13) can be expressed as

$$\sum_{j=1}^l a_j \left(\frac{z - x'_j}{z} \right),$$

where, for $j = 1, 2, \dots, l$, we have

$$a_j = \sum_{i=n_0+n_1+\dots+n_{j-1}+1}^{n_0+n_1+\dots+n_j} \left(\Phi\left(\frac{i}{n}\right) - \Phi\left(\frac{i-1}{n}\right) \right).$$

Consequently, (13) can be rewritten as

$$\begin{aligned} & \sum_{j=1}^l \left[\sum_{i=n_0+n_1+\dots+n_{j-1}+1}^{n_0+n_1+\dots+n_j} \left(\Phi\left(\frac{i}{n}\right) - \Phi\left(\frac{i-1}{n}\right) \right) \right] \left(\frac{z - x'_j}{z} \right) \\ &= \sum_{j=1}^l \left[\sum_{i=n_0+n_1+\dots+n_{j-1}+1}^{n_0+n_1+\dots+n_j} \left(\Phi\left(\frac{i}{n}\right) - \Phi\left(\frac{i-1}{n}\right) \right) \left(\frac{z - x'_j}{z} \right) \right]. \end{aligned} \quad (14)$$

Taking into account that

$$x'_j = x_i \text{ for } i = n_0 + n_1 + \dots + n_{j-1} + 1, \dots, n_0 + n_1 + \dots + n_j,$$

for every $j = 1, 2, \dots, l$, we have that (14) can be expressed as

$$\sum_{j=1}^l \left[\sum_{i=n_0+n_1+\dots+n_{j-1}+1}^{n_0+n_1+\dots+n_j} \left(\Phi\left(\frac{i}{n}\right) - \Phi\left(\frac{i-1}{n}\right) \right) \left(\frac{z-x_i}{z} \right) \right] \quad (15)$$

and using that $q = n_0 + n_1 + \dots + n_l$, it follows that (15) is the same as

$$\sum_{i=1}^q \left[\Phi\left(\frac{i}{n}\right) - \Phi\left(\frac{i-1}{n}\right) \right] \left(\frac{z-x_i}{z} \right).$$

■

4 Particular measures

In this section we show that the class C includes some poverty measures that are well-known from the literature. Since the formulation of some of these measures was originally given for finite populations, the representation (12) will be repeatedly used.

(i) By taking $\Phi(t) = t$ in (4) we obtain the “per-capita income gap” proposed by Foster, Greer and Thorbecke (1984). When X describes the income distribution of a finite population, (12) gives

$$I_X(\Phi, z) = \frac{1}{n} \sum_{i=1}^q \left(\frac{z-x_i}{z} \right).$$

(ii) Choosing the function

$$\Phi(t) = 1 - (1-t)^2, \quad t \in [0, 1],$$

we obtain the poverty measure

$$I_X(\Phi, z) = \int_0^1 2(1-t) \left(\frac{z - F_z^{-1}(t)}{z} \right) dt, \quad (16)$$

introduced by Thon (1979). If X describes the income distribution of a finite population, using (12) we see that (16) is the same as

$$I_X(\Phi, z) = \sum_{i=1}^q \frac{2n-2i+1}{n^2} \left(\frac{z-x_i}{z} \right), \quad (17)$$

which is the modified-Sen poverty index proposed by Shorrocks (1995).

(iii) More generally, a subclass $S \subset C$ of particular interest emerges from considering the weight function

$$\Phi_n(p) = \{1 - (1 - p)^n\}, \quad n \geq 1. \quad (18)$$

As noted by Duclos (2000) and Duclos and Grégoire (2002), $I_X(\Phi_n, z)$, $n \geq 2$, depends upon an ethical parameter n , which captures the sensitivity of poverty measurement to “exclusion” or “relative deprivation” aversion: the greater the value of n , the more weight is given to the relative deprivation of the poor. They refer to $I_X(\Phi_n, z) = S_X(n, z)$ as the equally distributed equivalent (EDE) poverty gap that is socially equivalent to the actual distribution of poverty gaps. $S_X(n, z)$ also can be interpreted as the higher poverty gap in a sample of n randomly selected poor individuals.

(iv) The choice

$$\Phi(t) = \frac{c^2}{4(c-1)} - \frac{1}{c-1} \left(\frac{c}{2} - t\right)^2, \quad t \in [0, 1], \quad c > 2,$$

gives

$$I_X(\Phi, z) = \int_0^1 \frac{2}{c-1} \left(\frac{c}{2} - t\right) \left(\frac{z - F_z^{-1}(t)}{z}\right) dt,$$

which was introduced by Thon (1983). In this case, (12) gives

$$I_X(\Phi, z) = \sum_{i=1}^q \frac{cn - 2i + 1}{(c-1)n^2} \left(\frac{z - x_i}{z}\right).$$

(v) Hagenaars (1987) presented general classes of poverty measures by applying the well-known income inequality measures of Dalton (1920) and Atkinson (1970) to poverty measurement. In particular, Hagenaars considered indices of the form

$$D^w = 1 - \frac{\sum_{i=1}^n w_i x_i^*}{z \sum_{i=1}^n w_i} = \sum_{i=1}^n W_i \left(\frac{z - x_i^*}{z}\right) \quad (19)$$

where $\sum_{i=1}^n W_i = 1$, n is the size of the total population and x_i^* , ordered such that $x_i^* \leq x_{i+1}^*$ for all i , are the values of the censored variable X_z^* .

These measures can be expressed according to (12) considering the weights $W_i = \Phi(i/n) - \Phi((i-1)/n)$. The condition $\sum_{i=1}^n W_i = 1$ is equivalent to $\Phi(1) - \Phi(0) = 1$, which holds from the fact that $\Phi(t)$ is a distribution function with support in $[0, 1]$.

5 Consistency with the axioms

Following Zheng (2000), for a poverty measure P the reasonable axioms include: (i) focus: P is not affected by changes in nonpoor incomes; (ii) symmetry: P is not affected if two people switch their incomes; (iii) replication invariance: P is not affected by the pooling of several identical populations (P is not affected by a m -replication), (iv) monotonicity: P increases if a poor person's income decreases; (v) strong transfer: P increases if income is transferred from a poor person to someone richer (may or may not be poor); (vi) weak transfer sensitivity: the increase in P due to a regressive transfer (from a poor person to someone richer) within the poor is inversely related to the income levels of the donor (no one crosses the poverty line as a result of the transfer); (vii) increasing poverty line: P is an increasing function of the poverty line.

In this section, we examine the properties of the measures of C in terms of their consistency with the above axioms. The consistency with the three first ones is very easy to show.

- (i) *Focus*: $I_X(\Phi, z)$ is not affected by changes in nonpoor incomes because it is based on the distribution of the censored variable at z , X_z^* .
- (ii) *Symmetry*: $I_X(\Phi, z)$ is not affected if two people switch their incomes because of the use of quantiles in the expression (4).
- (iii) *Replication invariance*: we must show that $I_X(\Phi, z)$ is not affected by a m -replication (which makes sense only for finite populations). This axiom makes the comparison of poverty levels between any two different-sized finite income distributions possible. First, we present the definition of a m -replication:

Definition 5 *Let X be a discrete variable which describes the income distribution of a population with size $n(x)$ and consider that it is represented by the vector $x = (x_1 \leq x_2 \leq \dots \leq x_{n(x)})$. Given a positive integer m , X' is obtained from X by a m -replication if $n(x') = m \cdot n(x)$ and $x' = (\underbrace{x, x, \dots, x}_m)$.*

Proposition 6 *Let X be a non-negative random variable which describes the income distribution of a finite population and let $z > 0$ be the poverty line. If X' is a m -replication of X then $I_{X'}(\Phi, z) = I_X(\Phi, z)$.*

Proof. Consider a discrete variable which describes the income distribution of a population with size $n(x)$. Suppose that we have α different incomes, $x_1 < x_2 < \dots < x_\alpha$, with respective mass $\frac{n_1}{n}, \frac{n_2}{n}, \dots, \frac{n_\alpha}{n}$, where $n(x) = \sum_{i=0}^{\alpha} n_i$, $n_0 = 0$. Let $z > 0$ be a poverty line and $l(x)$ integer such that $x_{l(x)} < z \leq x_{l(x)+1}$. If X' is a m -replication of X then $l(x') = l(x)$ and, from Theorem 3, we have

$$I_{X'}(\Phi, z) = \sum_{k=1}^{l(x')} \left(\Phi \left(\sum_{i=0}^k \frac{m n_i}{m n} \right) - \Phi \left(\sum_{i=0}^{k-1} \frac{m n_i}{m n} \right) \right) \left(\frac{z - x_k}{z} \right) = I_X(\Phi, z),$$

which proves that $I_X(\Phi, z)$ is not affected by a m -replication. ■

(iv) *Increasing poverty line*: the following result shows that this axiom is satisfied by members of C with strictly increasing Φ .

Proposition 7 *Let F be the distribution function of a non-negative random variable X and let z and z' denote poverty lines such that $z < z'$. If Φ is strictly increasing then $I_X(\Phi, z') > I_X(\Phi, z)$.*

Proof. Let z and z' be poverty lines such that $0 < z < z'$. It follows that

$$r_z = \sup\{F(x) / x < z\} \leq \sup\{F(x) / x < z'\} = r_{z'}.$$

Since $z < z'$, using (5) easily follows that

$$1 - H_{z'}(t) > 1 - H_z(t), \text{ for all } t \in [0, r_{z'}) \text{ such that } F^{-1}(t) \neq 0. \quad (20)$$

Therefore, using $r_z \leq r_{z'}$ together with (20) and $d\Phi > 0$, we have that

$$I_X(\Phi, z') = \int_0^{r_{z'}} (1 - H_{z'}(t)) d\Phi(t) > \int_0^{r_z} (1 - H_z(t)) d\Phi(t) = I_X(\Phi, z).$$

In the case where $F^{-1}(t) = 0$ for all $t \in [0, r_{z'})$, it follows from Proposition 1 that $I_X(\Phi, z') = \Phi(r_{z'})$ and $I_X(\Phi, z) = \Phi(r_z)$ and from the strictly increasing of Φ we obtain

$$I_X(\Phi, z') = \Phi(r_{z'}) > \Phi(r_z) = I_X(\Phi, z).$$

■

Remark 8 When Φ is not strictly increasing then the axiom is satisfied in a weak sense: for $z < z'$ we have $I_X(\Phi, z') \geq I_X(\Phi, z)$.

(v) *Monotonicity*: $I_X(\Phi, z)$ should increase if a poor person's income decreases (this axiom is related to the Pareto principle frequently used for social welfare comparisons). In the case of a finite population, the consistency with this axiom easily follows from a result of Hagenaars (1987). This author showed that an index of the form (19) satisfies the monotonicity axiom if $W_i > 0$, for all i . Since, in the case of a finite population, $I_X(\Phi, z)$ can be written in the form (19) with $W_i = \Phi(i/n) - \Phi((i-1)/n)$, it follows from Hagenaars' result that $I_X(\Phi, z)$ satisfies the monotonicity axiom if Φ is strictly increasing. The consistency in the continuous case is given in the following proposition.

Proposition 9 Let F be the distribution function of a non-negative random variable X , let $z > 0$ be the poverty line and let $I_X(\Phi, z)$ be a poverty measure in C with Φ strictly increasing. Then $I_X(\Phi; z)$ verifies the monotonicity axiom.

Proof. Let $z > 0$ be a poverty line and let F be the distribution function of the random variable X . Consider a random variable Y with distribution function G obtained from X when a small positive quantity of income, $\Delta > 0$, is taken from a tiny fraction, dt , of the population at the t^{th} percentile of the distribution, for $t < r_z$ (observe, in particular, that $r_z^X = r_z^Y = r_z$ but X_z^* and Y_z^* do not have the same distribution). This transformation is equivalent to move a small density of probability lower down the t^{th} percentile. So, it can be easily seen that

$$F(u) \leq G(u), \text{ for all } 0 \leq u \leq z,$$

which is equivalent to

$$F^{-1}(t) \geq G^{-1}(t) \text{ for all } 0 \leq t \leq r_z.$$

This means that $H_{X_z} \leq_{st} H_{Y_z}$ where \leq_{st} is the usual stochastic order. It follows from (1.A.7) in Shaked and Shanthikumar (2007) that

$$\int_0^{r_z} \Phi(t) dH_{X_z}(t) \leq \int_0^{r_z} \Phi(t) dH_{Y_z}(t) \text{ for all increasing } \Phi \quad (21)$$

and from (10) and (21) we get

$$I_X(\Phi, z) \leq I_Y(\Phi, z) \text{ for all increasing functions } \Phi.$$

In order to see that the inequality is strict, suppose that $I_X(\Phi, z) = I_Y(\Phi, z)$. Since Φ is strictly increasing, it follows from Theorem 1.A.8 of Shaked and Shanthikumar (2007) that X_z^* and Y_z^* have the same distribution, which is false by construction of Y . Consequently, $I_X(\Phi, z) < I_Y(\Phi, z)$ and the proof is complete. ■

Remark 10 *When Φ is not strictly increasing then the axiom is satisfied in a weak sense, i.e., $I_X(\Phi, z) \leq I_Y(\Phi, z)$.*

(vi) *Strong transfer*: $I_X(\Phi, z)$ should increase if income is transferred from a poor person to someone richer (may or may not be poor). Poverty indices which verify this axiom have a greater ethical preference for equality of income. For instance, all other things being the same, the more equal the distribution of income among the poor, the lower the level of poverty. As in the previous axiom, the consistency in the case of a finite population follows from a result of Hagenaars (1987). This author showed that an index of the form (19) satisfies the strong transfer axiom if $W_{i+1} < W_i$, for all i and $z \sum_{i=1}^n W_i$ does not depend on the proportion of the poor. Since, in the case of a finite population, $I_X(\Phi, z)$ can be written in the form (19) with $W_i = \Phi(i/n) - \Phi((i-1)/n)$, it follows from Hagenaars' result that $I_X(\Phi, z)$ satisfies the strong transfer axiom if Φ is strictly concave (observe that $z \sum_{i=1}^n W_i = z$ does not depend on the proportion of the poor).

In order to study the consistency with this axiom in the continuous case, we need the following result.

Lemma 11 *(Zygmund, 1959) If Φ is a nondecreasing function on $[0, \infty]$, finite and strictly concave on $(0, \infty)$, with $\Phi(0) = 0$, then*

$$\Phi(t) = \int_0^t \varphi(u) du, \quad t \in [0, \infty),$$

for some finite non-negative, strictly decreasing function φ on $[0, \infty)$.

Proposition 12 *Let F be the distribution function of a non-negative random variable X , let $z > 0$ be the poverty line and let $I_X(\Phi, z)$ be a poverty measure in C with Φ strictly increasing and strictly concave. Then $I_X(\Phi, z)$ verifies the strong transfer axiom.*

Proof. Suppose that Φ is a strictly concave distribution function. From Lemma 11 there exists a strictly decreasing, non-negative and integrable function, φ , such that

$$\Phi(t) = \int_0^t \varphi(u) du, \quad t \in [0, 1) \quad (22)$$

and (4) can be rewritten as

$$\int_0^1 \left(\frac{z - F_z^{-1}(t)}{z} \right) \varphi(t) dt. \quad (23)$$

Let $dI_{t,t+\delta}(\Delta)$ denote the increase in poverty due to a small transfer of income, $\Delta > 0$, from a tiny fraction, dt , of the poor population at the t^{th} percentile of the distribution to richer individuals at the $(t + \delta)^{th}$ percentile ($\delta > 0$). The strong transfer axiom will be satisfied if $dI_{t,t+\delta}(\Delta) > 0$ for all $t < r_z$ (observe that, at least, donors have to be poor). If the recipients are also poor, we have from (23) that

$$\begin{aligned} dI_{t,t+\delta}(\Delta) &= \frac{(z - F_z^{-1}(t + \delta) - \Delta)}{z} \varphi(t + \delta) dt - \frac{(z - F_z^{-1}(t + \delta))}{z} \varphi(t + \delta) dt \\ &\quad + \frac{(z - F_z^{-1}(t) + \Delta)}{z} \varphi(t) dt - \frac{(z - F_z^{-1}(t))}{z} \varphi(t) dt \\ &= \frac{\Delta}{z} [\varphi(t) - \varphi(t + \delta)] dt. \end{aligned} \quad (24)$$

Since φ is strictly decreasing it follows that $dI_{t,t+\delta}(\Delta) > 0$, as desired. Observe that if the recipients are not poor, we have

$$dI_{t,t+\delta}(\Delta) = \frac{\Delta}{z} \varphi(t) dt,$$

which is also strictly positive because Φ is strictly increasing and $\varphi = d\Phi$. ■

Remark 13 When Φ is not strictly concave then the axiom is satisfied in a weak sense, i.e., $dI_{t,t+\delta}(\Delta) \geq 0$.

(vii) *Weak transfer sensitivity:* the increase in $I_X(\Phi, z)$ due to a regressive transfer (from a poor person to someone richer) within the poor should be inversely related to the income levels of the donor; no one crosses the

poverty line as a result of the transfer. The idea of this axiom is that poverty indices should give more emphasis to transfers taking place lower down in the distribution, other things being equal. The consistency with this axiom in the case of a finite population also follows from a result of Hagenaars (1987). This author showed that an index of the form (19) satisfies the weak transfer sensitivity axiom if the difference $W_i - W_{i+1}$ is decreasing in i . Since, in the case of a finite population, $I_X(\Phi, z)$ can be written in the form (19) with $W_i = \Phi(i/n) - \Phi((i-1)/n)$, it follows from Hagenaars' result that if Φ is strictly concave and differentiable almost everywhere (a.e.) with strictly convex derivative (which implies that $W_i - W_{i+1}$ is decreasing in i) then $I_X(\Phi, z)$ satisfies the weak transfer sensitivity axiom.

In order to study the consistency with this axiom in the continuous case, we first observe that for a concave function Φ , its derivative, Φ' , exists (except possibly at a countable number of points). We will consider the poverty measures (4), where Φ is strictly concave and differentiable almost everywhere (a.e.), with $\Phi' = \varphi$, and φ is strictly convex.

Proposition 14 *Let F be the distribution function of a non-negative random variable X , let $z > 0$ be the poverty line and let $I_X(\Phi, z)$ be a poverty measure in C with Φ strictly concave and differentiable almost everywhere (a.e.) with strictly convex derivative. Then $I_X(\Phi, z)$ satisfies the weak transfer sensitivity axiom.*

Proof. Recall that $dI_{t,t+\delta}(\Delta)$ denotes the increase in poverty due to a small positive transfer of income, $\Delta > 0$, from a tiny fraction, dt , of the population at the t^{th} percentile of the distribution ($t < r_z$) to richer individuals at the $(t+\delta)^{th}$ percentile, $\delta > 0$. The weak transfer sensitivity axiom will be satisfied if $dI_{t,t+\delta}(\Delta)$ is decreasing in t (for $t < r_z$). Suppose that Φ is a strictly concave distribution function and that $\varphi = \Phi'$. Consider the expression of $dI_{t,t+\delta}(\Delta)$ given in equality (24),

$$dI_{t,t+\delta}(\Delta) = \frac{\Delta}{z}[\varphi(t) - \varphi(t+\delta)]dt$$

Since Φ is strictly concave then φ is strictly decreasing. In addition, φ is a strictly convex function by assumption. It follows that $dI_{t,t+\delta}(\Delta)$ is decreasing in t and $I_X(\Phi, z)$ satisfies the weak transfer sensitivity axiom. ■

Before ending this section, we will show that $I_X(\Phi, z)$ is not affected if we multiply income and poverty line by a common factor $a > 0$. This means

that the poverty measure is unaffected by either the unit or the currency against which income is measured. This property is not a basic axiom of poverty measures: it only identifies the special class of them called “relative measures”.

Proposition 15 *Let F be the distribution function of a non-negative random variable X and let $z > 0$ be the poverty line. Let G be the distribution function of the variable $Y = aX$, with $a > 0$. Then $I_X(\Phi, z) = I_Y(\Phi, az)$.*

Proof. Let F be the distribution function of a non-negative random variable, X , and consider a fixed poverty line $z > 0$. Let $G(y) = F(y/a)$ denote the distribution function of $Y = aX$, $G^{-1}(t) = aF^{-1}(t)$ the corresponding income quantile function and

$$r_{az}^Y = \sup \{G(y) / y < az\} = \sup \left\{ F(y/a) / \frac{y}{a} < z \right\} = r_z^X,$$

the corresponding poverty lines. It follows that $G_{az}^{-1}(t) = aF_z^{-1}(t)$ and therefore,

$$\begin{aligned} I_Y(\Phi, az) &= \int_0^1 \left(\frac{az - G_{az}^{-1}(t)}{az} \right) d\Phi(t) \\ &= \int_0^{r_{az}^Y} \left(\frac{az - G_{az}^{-1}(t)}{az} \right) d\Phi(t) = \int_0^{r_z^X} \left(\frac{az - aF_z^{-1}(t)}{az} \right) d\Phi(t) \\ &= \int_0^1 \left(\frac{z - F_z^{-1}(t)}{z} \right) d\Phi(t) = I_X(\Phi, z), \end{aligned}$$

and the proof is completed. ■

6 Poverty measures and TIP curves

There is a closed relationship between the class C of poverty measures and the TIP curve. Given a poverty line $z > 0$ and an income random variable X with cumulative distribution function F , the TIP (Three ‘I’s of Poverty) curve associated to X is given by

$$G_X(p, z) = \int_0^p (z - F_z^{-1}(t)) dt, \quad 0 \leq p \leq 1. \quad (25)$$

For a given population share $p \in [0, 1]$, this curve accumulates the largest $100p\%$ of poverty gaps. The TIP curve was introduced by Spencer and Fisher (1992), who called it absolute rotated Lorenz curve and it is also known as poverty gap profile (Shorrocks 1995) and cumulative poverty gap (CPG) curve (Davidson and Duclos, 2000). Jenkins and Lambert (1997, 1998a, 1998b) further elaborated this device and noticed that the curve exhibits three aspects of poverty of a distribution: incidence, intensity and inequality (the three ‘T’s of poverty). As shown by Duclos and Araar (2006) and Sordo, Ramos and Ramos (2007), orderings of income distributions by non-intersecting TIP curves correspond to unanimous orderings according to the subclass

$$C_1 = \{I(\Phi, z) \in C \text{ with } \Phi \text{ concave}\}.$$

When TIP curves intersect (which is often the case), we can still obtain unanimous orderings according to the subclass of C_1

$$C_2 = \{I(\Phi, z) \in C_1 \text{ such that } \phi \text{ is convex, where } \Phi'(t) = \phi(t)\},$$

by comparing TIP areas (see Sordo, Ramos and Ramos, 2007), that is, by comparing curves of the form

$$A_X(p, z) = \int_0^p G_X(t, z) dt, \quad 0 \leq p \leq 1.$$

From the results in Section 5, an index $I(\Phi, z) \in C_1$ satisfies the following axioms: focus, symmetry, replication invariance, scale invariance, monotonicity and transfer axiom. If, in addition, $I(\Phi, z) \in C_2$, then it also satisfies the weak transfer sensitivity axiom. Thus, a decision-maker who employs $I(\Phi, z) \in C_2$ is more sensitive to transfers occurring within the lower part of the distribution.

The proof of the following result can be found in Sordo, Ramos and Ramos² (2007).

Theorem 16 *Let X and Y be two non-negative income random variables and let $z > 0$ be a common poverty line. Then,*

(a) $\int_0^p G_X(t, z) dt \leq \int_0^p G_Y(t, z) dt$ for all $p \in [0, 1] \Leftrightarrow I_X(\Phi, z) \leq I_Y(\Phi, z)$, for

²For two different poverty lines $z_1, z_2 > 0$, this theorem should be rewritten in terms of the normalized TIP curves $\frac{1}{z_1}G_X(p, z_1)$ and $\frac{1}{z_2}G_Y(p, z_2)$

all $I(\Phi, z) \in C_1$.

(b) $A_X(p, z) dt \leq A_Y(p, z)$ for all $p \in [0, 1]$ and $G_X(1, z) dt \leq G_Y(1, z) \Leftrightarrow I_X(\Phi, z) \leq I_Y(\Phi, z)$, for all $I(\Phi, z) \in C_2$.

Part (a) of Theorem 16 has an important precedent in Jenkins and Lambert (1998a, 1998b). These authors gave a similar characterization connecting the non-intersection of TIP curves with unanimous orderings according to a class of poverty indices which are increasing schur-convex functions of absolute poverty gaps. Moreover, they obtained unambiguous results even when TIP curves cross once. However, they do not provide results for the case where TIP curves present multiple crossings. In these situations, poverty comparisons based on measures of C may become especially useful: part (b) of Theorem 16 suggests a comparison principle which can be used when TIP curves intersect more than once. The purpose of the following section is to illustrate with real data how to apply this comparison principle.

7 An illustration with Spanish data

To illustrate Theorem 16 we include a comparative study between poverty of several autonomous communities in Spain. We use Spanish data drawn from the Survey of Living Conditions (Encuesta de Condiciones de Vida, ECV 2009) conducted by the Instituto Nacional de Estadística for three communities: Andalusia, Extremadura and Murcia (these communities were selected because of their apparent similarity with respect to the distribution of disposable income). The ECV survey is harmonized with the statistical source for the European Union, Statistics on Income and Living Conditions (EU-SILC). We employ the variable “total disposable income of the household”, adjusted to take into account that we are dealing with individuals who are members of households of different size and composition (we make this adjustment employing the modified OECD equivalence scale). The unit of analysis chosen is the individual; the income assigned to each individual is the total income of the household to which they belong, adjusted according to the equivalence scale chosen in each case to ensure comparability. The value used for the poverty line is 60 per cent of the 2007 Spanish median disposable income per unit of consumption.

For the three communities we calculate the TIP curves for the year 2007

with the national common poverty threshold³. In particular, the curves obtained when comparing Extremadura and Murcia with Andalusia allow us to apply Theorem 16. TIP curves have been calculated using the blowing up factors provided by the INE due to the small sample size of certain Autonomous Communities (this factors are used by INE to obtain population rather than sample statistics). First, the TIP curves for Andalusia and Extremadura are compared and shown in Figure 1.

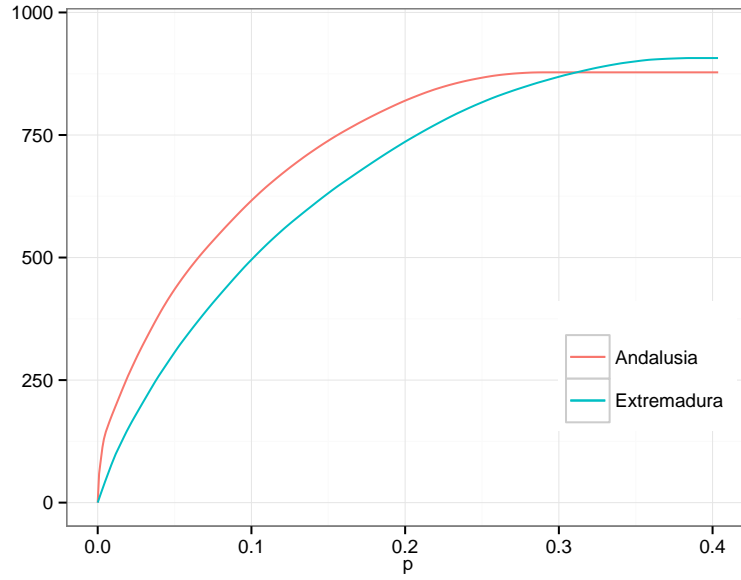


Figure 1: TIP curves for Andalusia and Extremadura. National common threshold of 2007. Modified OECD scale.

As can be observed in Figure 1, the curve TIP_{Ext} intersects the curve TIP_{And} from below. Thus, we consider the possibility of using the weaker criterion stated in Theorem 16(b) based on comparing TIP areas. In other words, with z fixed, the curves $A(p, z)$, with $0 \leq p \leq 1$, are constructed for both the Community of Andalusia and the Community of Extremadura, and they are shown in Figure 2.

As can be observed in Figure 2 we have the following order relationship:

$$A_{Ext}(p, z) \leq A_{And}(p, z)$$

³For some previous studies of poverty in Spain using TIP curves, see Del Río and Ruiz Castillo (2001a, 2001b) and Domínguez and Núñez (2006, 2007).

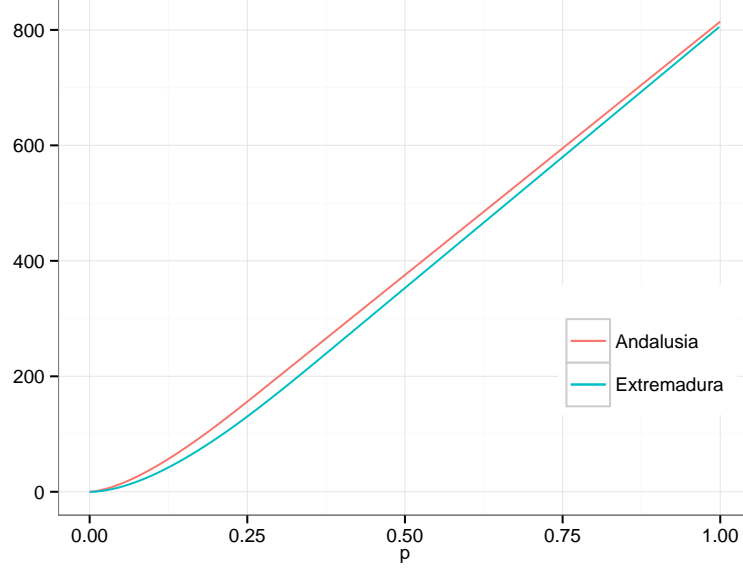


Figure 2: $A(p, z)$ curves for Andalusia and Extremadura. National common threshold of 2007. Modified OECD scale.

for each p . This leads us to conclude that poverty in Andalusia in 2007 is greater than or equal to that in Extremadura, for all the indices in the family C_2 and for the poverty threshold considered, as a result of the application of Theorem 16(b).

A similar study conducted to compare the Communities of Andalusia and Murcia reveals that the curves TIP_{And} and TIP_{Mur} cross each other twice (see Figure 3.) Then, we should use the weaker criterion based on comparing TIP areas.

With z fixed, the curves $A(p, z)$ with $0 \leq p \leq 1$, are constructed for both the Community of Andalusia and the Community of Murcia, and they are shown in Figure 4. The following conditions are also met:

$$G_{And}(1, z) > G_{Mur}(1, z), \quad (26)$$

and

$$G_{And}^{[2]}(p, z) \geq G_{Mur}^{[2]}(p, z) \quad (27)$$

for each p , with z being the national common threshold of 2007 (see Figures 3 and 4). Using Theorem 16(b) again, from (26) and (27) it follows that

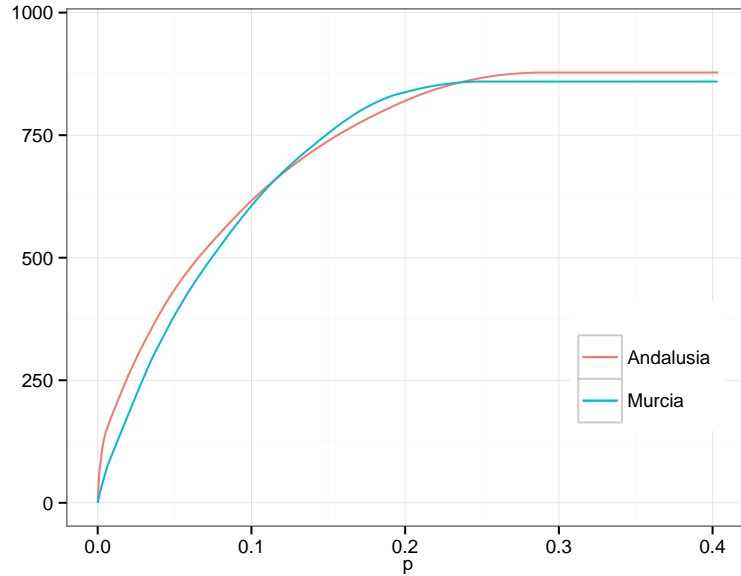


Figure 3: TIP curves for Andalusia and Murcia. National common threshold of 2007. Modified OECD scale.

poverty in Andalusia is greater than or equal to that in Murcia for all the indices of poverty $I \in C_2$, and for the poverty threshold considered.

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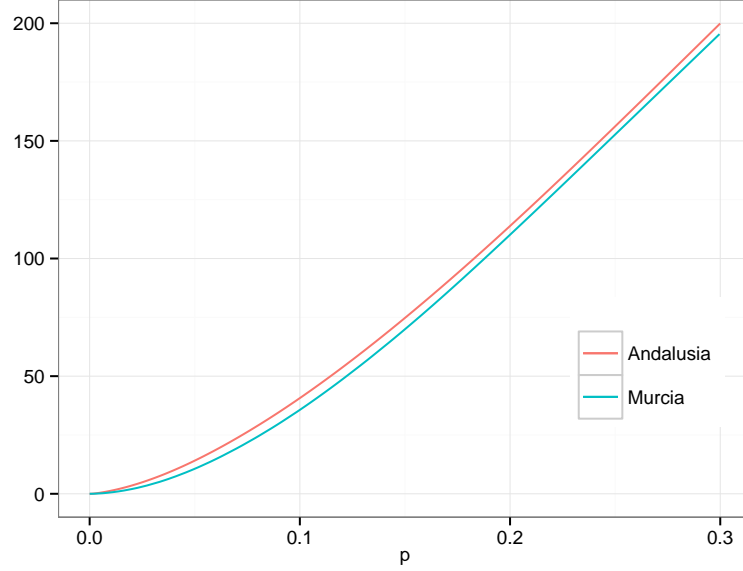


Figure 4: $A(p, z)$ curves for Andalusia and Murcia. National common threshold of 2007. Modified OECD scale.

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